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INFERENCE FOR STATIONARY RANDOM FIELDS GIVEN POISSON
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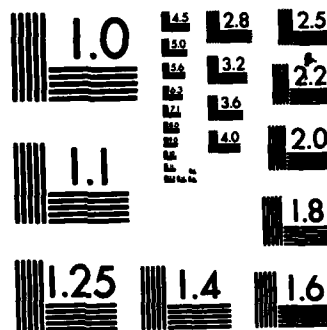
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DEPARTMENT OF MATHEMATICAL SCIENCES
THE JOHNS HOPKINS UNIVERSITY
BALTIMORE, MARYLAND 21218

Inference for Stationary Random Fields given Poisson Samples*

Alan F. Karr

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ABSTRACT

Given a d-dimensional random field and a Poisson process independent of it, suppose that it is possible to observe only the location of each point of the Poisson process and the value of the random field at that (randomly located) point. Nonparametric estimators of the mean and covariance function of the random field - based on observation over compact sets of single realizations of the Poisson samples - are constructed. Under fairly mild conditions these estimators are consistent (in various senses) as the set of observation becomes unbounded in a suitable manner. The state estimation problem of minimum mean squared error reconstruction of unobserved values of the random field is also examined.

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KEY WORDS AND PHRASES: random field, Poisson process, Poisson samples, nonparametric covariance estimation, state estimation
AMS (1980) SUBJECT CLASSIFICATIONS: 60G10, 60G55, 62M09, 62M20, 93E10

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1. Introduction. ^{over} In this paper ~~we~~ examine some questions of statistical inference -- specifically, estimation of the mean and covariance function,

as well as linear state estimation -- for stationary random fields observable only at the points of a (likewise stationary) Poisson process. Our main

results provide rather general conditions under which natural, easily

computable estimators are mean square consistent and asymptotically normal.

Our setting, more precisely, is this: let $Y = (Y_x)_{x \in \mathbb{R}^d}$ be an L^2 -

stationary random field on \mathbb{R}^d , where the dimension d is arbitrary but should

be thought of as two or greater (our results apply when $d = 1$ but fail to

exploit special structure of this case); we assume throughout that Y is

real-valued and continuous in probability and that $E[Y_x^2] < \infty$ for each x . Let

$N = \sum_{x_1} \delta_{x_1}$ be a stationary Poisson process on \mathbb{R}^d (see, e.g., Karr, 1985),

independent of Y ; this latter assumption is in force for the remainder of

the paper. While the point process N is completely observable, the random

field Y , the principal object of interest, is not; indeed, the only observations

available concerning it are its values $Y(x_1)$ at the points x_1 of N . Mathematically

it is convenient to describe the observations via the marked point process

$$(1.1) \quad \tilde{N} = \sum \delta_{(x_1, Y(x_1))}$$

on $\mathbb{R}^d \times \mathbb{R}$, in which each point of \tilde{N} is "marked" with the value of Y at that

point. In reality the stochastic system comprising N and Y is (even regardless

of partial observability) observable only over compact subsets K (taken for

technical reasons to be convex), leading to data described by σ -algebras

$$(1.2) \quad H(K) = \tilde{F}^N(K \times \mathbb{R}),$$

where $\tilde{F}^N(A) = \sigma(\tilde{N}(B) : B \subset A)$ corresponds to complete observation of \tilde{N}

over the set A . Just a single realization of \tilde{N} is observed (rather than

i.i.d. copies); our limit theorems pertain to asymptotics as K increases

to \mathbb{R}^d .

The statistical estimation problems we treat are estimation of the mean $m = E[Y_x]$, possibly with the complication that the intensity ν of N (which satisfies $E[N(K)] = \nu(K)$, where λ denotes Lebesgue measure) is unknown, and estimation of the covariance function

$$(1.3) \quad R(x) = \text{Cov}(Y_y, Y_{y+x}), \quad x \in \mathbb{R}^d,$$

from observations $H(K)$. In addition we consider state estimation, i.e.,

minimum mean squared error reconstruction of unobserved values of the random

field Y ; this question is of evident physical importance in applications

such as precipitation (rainfall is observed only at rain gauges) and geophysics

(mineral or petroleum reserves are to be estimated from test drillings).

To remain consistent with the emphasis on second-moment aspects we restrict

attention to state estimators $\hat{Y}(x)$ that are (deterministic) linear functions of the observations, albeit in two different senses.

The most direct antecedent of this paper is Masry (1983), which presents similar techniques and results for $d = 1$; Masry (1978) treats the related issue of spectral estimation when $d = 1$, from the different perspective of "synchronous" data $Y(X_1), \dots, Y(X_n)$ (the X_i are now linearly ordered), which is more restrictive than ours (the X_i are not assumed observable)

but fails to generalize meaningfully to higher dimensions. Poisson sampling

of stationary processes on \mathbb{R} has been known at least since Shapiro/Silverman

(1960) to be superior to regular sampling because the former (but not the

latter) is alias-free, i.e., the law of the Poisson samples determines

that of the underlying process; our Theorem (2.1) below is a corresponding

uniqueness property. Kingman (1963) and Karr (1982, 1984, 1985) address

additional aspects of Poisson sampling of processes on \mathbb{R} .

Adler (1981) and Yaglom (1983) are two general sources concerning random fields; the latter, in particular, emphasizes spectral theory.

In this paper we do not impose the further (and common) stipulation that the random field be isotropic; presumably our techniques could be specialized and our results sharpened were this to be done.

The remainder of the paper is organized in the following manner.

Section 2 contains but a single result: a uniqueness theorem ensuring that inference is possible in principle given Poisson samples of a random field; in this theorem above our blanket assumptions concerning N and Y are relaxed considerably. Sections 3 and 4 treat estimation of the mean m and covariance function R , respectively; in each we devise estimators that are mean square consistent and asymptotically normal. Finally, linear state estimation is considered in Section 5.

Additional aspects: other kinds of partial observations, mixed forms of observation and the problem of combined statistical and state estimation, will be treated in a subsequent paper.

2. Uniqueness. In statistical parlance the property established in Theorem (2.1) is identifiability: the law of a random field Y is determined uniquely by that of the marked point process \bar{N} of (1.1), even under very little restriction (much less than stationarity) on Y or on the Poisson process N . (2.1) THEOREM. Let Y be a random field on R^d that is continuous in probability, let N be a Poisson process on R^d with diffuse mean measure μ satisfying $\mu(G) > 0$ for every open set G , assume that Y and N are independent, and let \bar{N} be the marked point process of (1.1). Then the law of \bar{N} determines that of Y .

PROOF. Let Y_1, Y_2 be random fields fulfilling the hypotheses of the theorem, with associated marked point processes \bar{N}_1, \bar{N}_2 , respectively, and suppose without loss of generality that $|Y_1| \leq 1$. Then for h a function on R^d with $0 < h < 1$ and $f(x, u) = \log(1 - uh(x))$, $|u| \leq 1$, we have

$$\begin{aligned} E(\exp(-\int h(x) Y_1(x) \mu(dx))) &= E(\exp(-\int [1 - e^{-\int h(x) Y_1(x)}] \mu(dx))) \\ &= E(E(\exp(-\int h(x) Y_1(x) N(dx)) | Y_1)) \\ &= E(\exp(-\int h d\bar{N}_1)) \\ &= E(\exp(-\int h(x) Y_2(x) \mu(dx))) ; \end{aligned}$$

the last equality is a reversal of the first three. Consequently by the uniqueness theorem for Laplace functionals of random measures (see for example Kallenberg, 1983, or Karr, 1985) the random measures $\mu_1(dx) = Y_1(x) \mu(dx)$ and $\mu_2(dx) = Y_2(x) \mu(dx)$ are identically distributed. Given x_1, \dots, x_k in R^d , for each i choose open sets G_{in} such that $G_{in} \ni x_i$. Then $\mu(G_{in}) > 0$ for each i and n , while $\mu(G_{in}) \rightarrow 0$ for each i . Thus by continuity in probability of Y_1 and Y_2 ,

$$\begin{aligned} (Y_1(x_1), \dots, Y_1(x_k)) &= \lim_{n \rightarrow \infty} (M_1(G_{1n})/\mu(G_{1n}), \dots, M_1(G_{kn})/\mu(G_{kn})) \\ &\stackrel{d}{=} \lim_{n \rightarrow \infty} (M_2(G_{1n})/\mu(G_{1n}), \dots, M_2(G_{kn})/\mu(G_{kn})) \\ &= (Y_2(x_1), \dots, Y_2(x_k)) ; \end{aligned}$$

the limits are in the sense of convergence in distribution. Hence $Y_1 \stackrel{d}{=} Y_2$. \square

3. Estimation of the mean. Now and for the remainder of the paper we suppose Y is L^2 -stationary with mean m and covariance function R (cf. (1.2)) and that N is a stationary Poisson process with intensity v . Our concern is estimation of m from observations $H(K)$ given by (1.3), where K is a compact, convex set in R^d . After treating the case that v is known we pass to the case that it is not.

We presume observation over a compact, convex set K , leading in case v is known to the $H(K)$ -estimator

$$(3.1) \quad \hat{m} = (vA(K))^{-1} \int_K Y dN ;$$

dependence on the "sample size" K is ordinarily suppressed. Evidently these estimators are unbiased. Their asymptotic properties ensue in part from the following preliminary result, in which $\delta(K)$ denotes the supremum of the radii of Euclidean balls contained in K ; its elementary analytical proof is omitted. Note that $\lambda(K) \rightarrow \infty$ as $\delta(K) \rightarrow \infty$.

(3.2) LEMMA. For each $y \in \mathbb{R}^d$, and with $K-y = \{x-y: x \in K\}$,

$$(3.3) \quad \lim_{\delta(K) \rightarrow \infty} \frac{\lambda(K \cap (K-y))}{\lambda(K)} = 1. \quad \square$$

Mean square consistency then follows easily.

(3.4) PROPOSITION. For each K ,

$$\text{Var}(\hat{a}) = \lambda(K)^{-1} \left[\frac{\lambda(K)^2}{v} + \int_K \lambda(y) \frac{\lambda(K \cap (K-y))}{\lambda(K)} dy \right],$$

consequently, if the covariance function λ is integrable, i.e.,

$$(3.6) \quad \int_{\mathbb{R}^d} \lambda(y) dy < \infty,$$

then $\mathbb{E}[(\hat{a}-a)^2] \rightarrow 0$ as $\delta(K) \rightarrow \infty$.

PROOF. In (3.5), (3.6) and elsewhere, Lebesgue measure is denoted also by dy . Only the property

$$(3.7) \quad \mathbb{E}[N(dx)N(dy)] = v^2 dx dy + v dx dy$$

of Poisson processes, together with straightforward calculations, is needed to derive (3.5); see also the proof of Proposition (3.12). Given that

(3.6) holds, mean square consistency follows from (3.5) by means of Lemma (3.2). \square

While Proposition (3.4) does not depend crucially on the Poisson nature of N - if N were simply a stationary point process one could replace (3.7) by the appropriate second moment measure (see Karr, 1985, or Krickeberg, 1982), which would have to be assumed locally finite - the following central limit theorem is much more dependent.

(3.8) PROPOSITION. Assume that the covariance function λ fulfills (3.6) and that as $\delta(K) \rightarrow \infty$,

$$\lambda(K)^{-1/2} \int_K (v(x)-a) dx \xrightarrow{d} N(0, \int_K v(y) dy).$$

Then $\lambda(K)^{1/2}(\hat{a}-a) \xrightarrow{d} N(0, \sigma^2)$, where

$$(3.10) \quad \sigma^2 = \int_K v(y) dy + v^{-1} \int_K (v(x)-a)^2 dx.$$

PROOF. Conditions implying (3.9) are given, e.g., in Yaglom (1983); the asymptotic variance is necessarily $\int_K v(y) dy$. For $a \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{E}[\exp\{i a \lambda(K)^{1/2}(\hat{a}-a)\}] \\ &= \mathbb{E}[\exp\{i a \lambda(K)^{1/2} \int_K (v(x)-a) dx - \lambda(K)^{-1/2} \int_K v(x) dx\}] \\ &= \mathbb{E}[\mathbb{E}[\exp\{i a \lambda(K)^{1/2} \int_K (v(x)-a) dx - \lambda(K)^{-1/2} \int_K v(x) dx\} | \mathcal{V}]] \\ &= \mathbb{E}[\exp\{i a \lambda(K)^{1/2} \int_K (v(x)-a) dx - \lambda(K)^{-1/2} \int_K v(x) dx\}] \\ &= \mathbb{E}[\exp\{i a \lambda(K)^{1/2} \int_K (v(x)-a) dx - (\alpha^2/2) \lambda(K) \int_K v^2 dx\}] \\ &= \exp\{-\frac{1}{2} \alpha^2 \lambda(K) \int_K v^2 dx - \frac{\alpha^2}{2} \lambda(K) \int_K v^2 dx\}, \end{aligned}$$

(within error converging in probability to zero)

which completes the proof. \square

When the intensity v is unknown, one simply replaces it in (3.1) by the obvious - recall that N is itself observable - estimator $N(K)/\lambda(K)$, leading to

$$(3.11) \quad \hat{a} = N(K)^{-1} \int_K v(x) dx,$$

which in fact has some properties superior to those of the estimator given by (3.1).

(3.12) PROPOSITION. Let \hat{a} be given by (3.11). Then \hat{a} is unbiased; furthermore

$$(3.13) \quad \text{Var}(\hat{m}) = \lambda(K)^{-1} / R(y) \frac{\lambda(K \cap (K-y))}{\lambda(K)} dy$$

$$+ (R(0) - \lambda(K))^{-1} / R(y) \frac{\lambda(K \cap (K-y))}{\lambda(K)} dy E[N(K)^{-1} 1_{\{N(K) > 0\}}],$$

so that if (3.6) holds then $E[(\hat{m} - m)^2] \rightarrow 0$ as $\delta(K) \rightarrow \infty$.

PROOF. That \hat{m} is unbiased follows from

$$E(\hat{m}) = E[E(\hat{m} | N)] = E[N(K)^{-1} \int_K m dN] = m,$$

here we have adapted the convention that $0/0 = 1$. In the same way,

$$\begin{aligned} E(\hat{m}^2) &= E[E(N(K)^{-2} \int_K \int_K v(x) N(dx) \int_K v(y) N(dy) | N)] \\ &= E[N(K)^{-2} \int_K \int_K (m^2 + R(x-y)) N(dx) N(dy)], \end{aligned}$$

which can be evaluated using the conditional uniformity property of N to produce (3.13); we omit the computational details. \square

The term $E[N(K)^{-1} 1_{\{N(K) > 0\}}]$ can be calculated in more detail:

$$E[N(K)^{-1} 1_{\{N(K) > 0\}}] = (\nu \lambda(K))^{-1} + \int_0^\infty e^{-t} (\nu \lambda(K) - t)^{-2} dt.$$

Note that in some circumstances the variance of the estimator \hat{m} of (3.11) is less than that of the estimator (3.1); hence the former may be preferable even when ν is known.

Since $N(K)/\nu \lambda(K) \rightarrow 1$ almost surely as $\delta(K) \rightarrow \infty$, it follows that asymptotic normality obtains under the hypotheses of Proposition (3.8), and with the same asymptotic variance.

(3.14) PROPOSITION. Let \hat{m} be given by (3.11) and suppose that the hypotheses of Proposition (3.8) are fulfilled. Then as $\delta(K) \rightarrow \infty$, $\lambda(K)^{1/2} (\hat{m} - m) \xrightarrow{d} N(0, \sigma^2)$, where σ^2 is given by (3.10). \square

4. Estimation of the covariance function. To simplify the notation and exposition we assume that ν is known and that m is known and equal to zero.

In view of Section 3 and the form of our estimators neither of these assumptions is restrictive. Our setting and approach are entirely non-parametric. We employ kernel estimators analogous to those in Masry (1983).

Specifically, let w be a positive, bounded, isotropic density function on \mathbb{R}^d , let q_K be positive constants (in practice depending only on $\delta(K)$) such that $q_K \rightarrow 0$ and $q_K^{-d} \lambda(K) \rightarrow \infty$ as $\delta(K) \rightarrow \infty$ (perhaps at a prescribed rate), let $v_K(x) = q_K^{-d} w(x/q_K)$ and finally let $N^{(2)}(dx_1, dx_2) = N(dx_1) N(dx_2) 1_{\{x_1 \neq x_2\}}$. Then, the estimator corresponding to observations $H(K)$ is

$$(4.1) \quad \hat{R}(K) = \frac{1}{\nu \lambda(K)} \int_K \int_K v_K(x - x_1 + x_2) v(x_1) v(x_2) N^{(2)}(dx_1, dx_2).$$

The interpretation is the usual: $w_K(x - x_1 + x_2)$ is an approximation to $1_{\{x - x_1 + x_2 = 0\}} = 1_{\{x = x_1 - x_2\}}$, and the properties are substantially similar to those established by Masry (1983) for the one-dimensional case.

To describe those properties we require the fourth order cumulant function

$$(4.2) \quad \begin{aligned} Q(x_1, x_2, x_3) &= E[v(0) v(x_1) v(x_2) v(x_3)] - R(x_1) R(x_2 - x_3) \\ &\quad - R(x_2) R(x_3 - x_1) - R(x_3) R(x_1 - x_2) \end{aligned}$$

of v .

(4.3) THEOREM. Assume that R is continuous and satisfies (3.6) and that the fourth-order cumulant function Q exists and satisfies

$$(4.4) \quad \sup_{x_1, x_2} \int_{\mathbb{R}^d} |Q(x + x_1, x, x_2)| dx < \infty.$$

Then

a) For each x ,

$$(4.5) \quad \hat{E}(\hat{R}(x)) = \int_{K - K} v(y) R(x - y) \frac{\lambda(K \cap (K - x + y))}{\lambda(K)} dy \rightarrow R(x)$$

as $\delta(K) \rightarrow \infty$.

b) For each x_1, x_2 ,

$$(4.6) \quad \lim_{\alpha(K) \rightarrow 0} \lambda(K) \alpha(K) \text{Cov}(\hat{R}(x_1), \hat{R}(x_2))$$

$$= (\nu^{-2} \int w(x)^2 dx) [Q(0, x_1, x_1) + 2R(x_1)^2 + R(0)^2] \lambda(x_1 - x_2).$$

PROOF. The first expression in (4.5) follows from (4.1) and the property

that

$$E\{N^{(2)}(dx_1, dx_2)\} = \nu^2 dx_1 dx_2$$

$$\text{(which is immediate from } E\{N(dx_1)N(dx_2)\} = \nu^2 dx_1 dx_2 \nu dx_1 \epsilon_{x_1}(dx_2)\text{)}$$

these give

$$E\{\hat{R}(x)\} = \lambda(K)^{-1} \int_K \int_K w(x-x_1+x_2) R(x_1-x_2) dx_1 dx_2,$$

which in turn yields (4.5). As $\delta(K) \rightarrow 0$,

$$\lambda(K \cap (K-x+q, y)) / \lambda(K) = \lambda(K \cap (K-x)) / \lambda(K) \rightarrow 1$$

by Lemma (3.2) and the assumption that $q_K \rightarrow 0$; consequently the convergence statement in (4.5) holds by the dominated convergence theorem.

Concerning b),

$$(4.7) \quad E\{\hat{R}(x_1)\hat{R}(x_2)\} = (\nu^2 \lambda(K))^{-2} \int_K \int_K w(x_1-x_1+x_2) w(x_2-y_1+y_2) \\ \times E\{N^{(2)}(x_1, x_2)N^{(2)}(y_1, y_2)\} \\ \times E\{N^{(2)}(dx_1, dx_2)N^{(2)}(dy_1, dy_2)\},$$

which is expanded as follows: first of all,

$$(4.8) \quad E\{N^{(2)}(x_1, x_2)N^{(2)}(y_1, y_2)\} = Q(x_2-x_1, y_1-x_1, y_2-x_1) + R(x_2-x_1)R(y_2-y_1) \\ + R(y_1-x_1)R(y_2-x_1) + R(y_2-x_1)R(x_2-y_1),$$

while in addition

$$\int_{K-K} w(x)^2 dx + \int_R d w(x)^2 dx$$

(the latter integral, incidentally, is finite because w is a bounded density function) but this is shown easily. \square

$$(4.9) \quad E\{N^{(2)}(dx_1, dx_2)N^{(2)}(dy_1, dy_2)\}$$

$$= \nu^4 dx_1 dx_2 dy_1 dy_2$$

$$+ \nu^3 dx_1 dx_2 \epsilon_{x_1}(dy_1) dy_2 + \nu^3 dx_1 dx_2 dy_1 \epsilon_{x_2}(dy_2)$$

$$+ \nu^3 dx_1 dx_2 \epsilon_{x_2}(dy_1) dy_2 + \nu^3 dx_1 dx_2 dy_1 \epsilon_{x_1}(dy_2)$$

$$+ \nu^2 dx_1 dx_2 \epsilon_{x_1}(dy_1) \epsilon_{x_2}(dy_2) + \nu^2 dx_1 dx_2 \epsilon_{x_2}(dy_1) \epsilon_{x_1}(dy_2).$$

substitution of (4.8) and (4.9) into (4.7) produces a lengthy expression that we do not reproduce in its entirety; the dominant contributions to each term, in view of the integrability hypothesis (4.4), arise from $\nu^2 dx_1 dx_2 \epsilon_{x_1}(dy_1) \epsilon_{x_2}(dy_2)$ and $\nu^2 dx_1 dx_2 \epsilon_{x_2}(dy_1) \epsilon_{x_1}(dy_2)$, so that within $O(\lambda(K)^{-1})$ we have

$$(4.10) \quad \text{Cov}(\hat{R}(x_1), \hat{R}(x_2))$$

$$= (\nu^2 \lambda(K))^{-1} \int_K \int_K w(x_1-x_2) (w_K(x_2-x) + w_K(x_2+x))$$

$$\times (Q(0, x, x) + 2R(x)^2 + R(0)^2) \frac{\lambda(K \cap (K-x))}{\lambda(K)} dx$$

$$= (\alpha_K^{-2} \nu^2 \lambda(K))^{-1} \int_{K-K} w(x) (w(x + \alpha_K^{-1}(x_2-x_1)) + w(x + \alpha_K^{-1}(x_2-x_1)))$$

$$\times (Q(0, x_1-x_2, x_1-x_2) + 2R(x_1-x_2)^2 + R(0)^2)$$

$$\times \frac{\lambda(K \cap (K-x_1+x_2))}{\lambda(K)} dx,$$

from which (4.6) follows by another application of Lemma (3.2) in conjunction with the remaining assumptions. It is also necessary to observe that as

$$\delta(K) \rightarrow 0$$

Hence in particular, $\text{Var}(\hat{R}(x)) \rightarrow 0$ for each x (recall that $\alpha_K^d(x) \rightarrow \infty$) and the estimators \hat{R} are mean square consistent pointwise in x . Moreover, for $x_1 \neq x_2$, $\hat{R}(x_1)$ and $\hat{R}(x_2)$ are asymptotically uncorrelated, while pleasant in some respects this conclusion is disappointing in others, for it precludes, e.g., a functional central limit theorem for the \hat{R} as random processes, since any Gaussian limit process would have independent values at distinct points and hence could not admit even a separable version.

Although the global error measure $E[\int_K (\hat{R}-R)^2]$ does not converge to zero its rate of growth can be ascertained rather precisely, at least under mild additional assumptions.

(4.11) THEOREM. Suppose that the hypotheses of Theorem (4.3) are satisfied, that the covariance function R is bounded and twice continuously differentiable, that

$$(4.12) \quad \int_K d|y|^2 w(y) dy < \infty$$

and that $\lambda(K)\alpha_K^{4+d} \rightarrow 0$. Then

$$(4.13) \quad \lim_{\delta(K) \rightarrow \infty} \alpha_K^d E[\int_K (\hat{R}(x) - R(x))^2 dx] = R(0)^2.$$

PROOF. From (4.10), for each x , within error $O(\lambda(K)^{-1})$

$$\text{Var}(\hat{R}(x)) = \frac{1}{\alpha_K^d(K)} \int_{K-x} w(z)^2 [Q(0, x-Q_K z, x-Q_K z) + 2R(x-Q_K z)^2 + R(0)^2] \\ \times \frac{\lambda(K \cap (K-x-Q_K z))}{\lambda(K)} dz,$$

consequently

$$(4.14) \quad \int_K \text{Var}(\hat{R}(x)) dx = \frac{1}{\alpha_K^d(K)} \int_{K-x} w(z)^2 dz \int_K (Q(0, x, x) + R(x)^2) dx + \lambda(K)R(0)^2 \\ + O\left(\frac{1}{\alpha_K^d(K)}\right) \\ = \frac{R(0)}{\alpha_K^d} + o(1).$$

The differentiability assumption on R and Taylor's theorem combine with (4.5) to yield (subscripts denote partial derivatives)

$$E[\hat{R}(x)] = \int_{K-x} w(z) [R(x) + \alpha_K \sum R_{1j}(x) z_{1j} + \frac{\alpha_K^2}{2} \sum R_{1j1j}(x) z_{1j}^2 + O(\alpha_K^3)] \\ \times \frac{\lambda(K \cap (K-x-Q_K z))}{\lambda(K)} dz \\ = R(x) \int_{K-x} w(z) \frac{\lambda(K \cap (K-x-Q_K z))}{\lambda(K)} dz + O(\alpha_K^2)$$

(the integrated first derivative terms vanish individually by isotropy of w , while the second derivative term is $O(\alpha_K^2)$ by (4.12))

$$= R(x) + O(\lambda(K)^{-1}) + O(\alpha_K^2).$$

Therefore

$$(4.15) \quad \int_K [E(\hat{R}(x)) - R(x)]^2 dx = O\left(\frac{1}{\lambda(K)} + \alpha_K^2 + \lambda(K)\alpha_K^d\right).$$

Together (4.14) and (4.15) imply (4.13). \square

Asymptotic normality of $\hat{R}(x)$ for individual values of x can be established under suitably stringent hypotheses; as observed previously, since $\text{Cov}(\hat{R}(x_1), \hat{R}(x_2)) \rightarrow 0$ for $x_1 \neq x_2$ there can be no corresponding functional central limit theorem.

(4.16) THEOREM. Assume that Y admits finite moments of all orders and that for every $k \geq 2$ the cumulant function of order k (cf. Brillinger, 1981) satisfies

$$(4.17) \quad \int_{\mathbb{R}^d} |Q^{(k)}(y_1, \dots, y_{k-1})| dy_1 \dots dy_{k-1} < \infty,$$

assume in addition that R is twice continuously differentiable and that $\int |y|^2 w(y) dy < \infty$. Then for each $x, (\lambda(K)Q_{\lambda}^{(k)})^{-1} [R(x) - R(x)] \xrightarrow{d} N(0, \sigma^2(x))$, where

$$(4.18) \quad \sigma^2(x) = (\sqrt{2} \int w(y)^2 dy) (Q(0, x, x) + 2R(x)^2 + R(0)^2).$$

PROOF. The argument used by Maury (1983) to prove Theorem 3.3 there applies with essentially only notational changes; it consists in application to the random measure $M(dx) = Y(x)N(dx)$ of the rules of Leonov/Shiryayev (1959) for computation of cumulants, in order to show that for each $k \geq 3$ the k th-order cumulant of $(\lambda(K)Q_{\lambda}^{(k)})^{-1} [R(x) - R(x)]$ behaves asymptotically as $(\lambda(K)Q_{\lambda}^{(k)})^{-1} \lambda^{-k/2}$ and so converges to zero. \square

We conclude the section with brief discussion of alternative estimators, to which various results concerning estimation for stationary point processes - see Jollivet (1981), Karr (1985) and Krickeberg (1982) - are germane. The signed measure

$$(4.19) \quad Q(\lambda) = \int_{\Lambda} R(x) dx$$

can be estimated as follows. Assume that Λ is bounded; then the estimator

$$(where $M(dx) = Y(x)N(dx)$)$$

$$(4.20) \quad \hat{Q}(\lambda) = (\sqrt{2} \lambda(K))^{-1} \int_{\Lambda} M(dx_1) \int_{\Lambda} M(dx_2) 1(x_2 - x_1 \in \lambda)$$

is $H(K \cup (\lambda + K))$ -measurable. Provided that $0 \notin \Lambda$ this estimator is unbiased. Moreover, strong (i.e., almost sure) consistency holds under minimal ergodicity hypotheses.

(4.21) THEOREM. Suppose that the random measure $N(dx) = Y(x)N(dx)$ is ergodic (every random variable $Y = H(N) \in \mathcal{F}^N$ invariant in the sense that $Y = H(N \tau_y^{-1})$ for every y , where τ_y is the translation $x \mapsto x+y$, is degenerate). Then for each set Λ with $0 \notin \Lambda$, almost surely $\hat{Q}(\lambda) \rightarrow Q(\lambda)$ as $\delta(\lambda) \rightarrow \infty$. PROOF. The spatial ergodic theorem of Nguyen/Zessin (1979, Proposition (4.23)) applies directly; see Karr (1985, Theorem 9.2) for more detailed description of that theorem. \square

Asymptotic normality follows from the central limit theorem of Jollivet (1981), whose ultimate basis -- the paper of Leonov/Shiryayev (1959) -- is the same as that of (4.16); however for the usual technical reasons arising in connection with measure-valued processes we require a smooth integrand in (4.19). Thus for f a continuous function with compact support, with

$$(4.19') \quad Q(f) = \int f dQ = \int R(x) f(x) dx,$$

let

$$(4.20') \quad \hat{Q}(f) = (\sqrt{2} \lambda(K))^{-1} \int_{\Lambda} M(dx_1) \int_{\Lambda} M(dx_2) f(x_2 - x_1).$$

Provided that $f(0) = 0$, $Q(f)$ is unbiased for each K (and computable from the observations $H(K \cup (\text{support}(f) + K))$); moreover Theorem (4.21) remains valid and in addition the following central limit theorem obtains.

(4.22) THEOREM. Suppose that M is ergodic, that (4.17) is fulfilled and that f is continuous with compact support not containing zero. Then $\lambda(K)^{1/2} [\hat{Q}(f) - Q(f)] \xrightarrow{d} N(0, \sigma^2(f))$, where $\sigma^2(f)$ is an appropriate variance, not calculated here. \square

Comparison of (4.20') with estimators in Karr (1985) and Krickeberg (1982) reveals that the $\hat{Q}(f)$ estimate the reduced second moment measure

of N , these references may be consulted for details and complements, including a functional version of Theorem (4.22).

5. Linear state estimation. Rather than estimation of unknown aspects of the probability law of Y our concern is now the state estimation problem of minimum mean squared error (MSE) reconstruction of unobserved portions of individual sample paths $x \rightarrow Y(x)$. This problem has a significant applications component: for example, mineral and oil reserves must be estimated and mapped from a small number of test drillings, and areal precipitation must be inferred from measurements at isolated raingages. We retain the entire overall structure, together with the assumptions that Y has known mean $m = 0$ and known covariance function R , leaving untreated the more difficult problem of state estimation when m or R is unknown.

For simplicity (and because of a dearth of more general results) we consider only linear state estimation, which we initially take to mean the following. Given observation of the marked point process \tilde{N} of (1.1) over a bounded set A in \mathbb{R}^d we seek that function h on $\mathbb{R}^d \times A$ for which the linear state estimator

$$(5.1) \quad \hat{Y}(x) = \int_A h(x, z) Y(z) N(dz)$$

minimizes the mean squared error $E\{\hat{Y}(x) - Y(x)\}^2$.

The solution is very similar to those of many related problems.

(5.2) PROPOSITION. With A fixed the optimal linear state estimator \hat{Y} corresponds to any function h^* satisfying the equation

$$(5.3) \quad R(x-y) = v \int_A h^*(x, z) R(y-z) dz + h^*(x, y) R(0)$$

for $x \in \mathbb{R}^d, y \in A$. For each x ,

$$(5.4) \quad E\{\hat{Y}(x) - Y(x)\}^2 = h^*(x, x) R(0).$$

PROOF. The following argument can be made more rigorous by putting it in integrated form, but is clearer as it stands. With x fixed, by Hilbert space theory the optimal linear state estimator $\hat{Y}(x)$ fulfills the normal equations

$$E\{\hat{Y}(x) - Y(x)\} Y(y) N(dy) = 0, \quad y \in A,$$

therefore h^* must satisfy

$$\begin{aligned} VR(x-y) dy &= E\{Y(x) Y(y) N(dy)\} \\ &= E\left\{\int_A h^*(x, z) Y(z) N(dz)\right\} Y(y) N(dy) \\ &= \int_A h^*(x, z) R(y-z) [v^2 dz dy + v c_y(z) dy], \end{aligned}$$

which is the same as (5.3).

Derivation of (5.4) from (5.3) is a straightforward calculation, given here in skeletal form:

$$\begin{aligned} E\{\hat{Y}(x) - Y(x)\}^2 &= E\left\{\int_A h^*(x, y) Y(y) N(dy) / h^*(x, z) N(dz)\right\} \\ &\quad - 2v / h^*(x, y) R(x-y) dy + R(0) \\ &= v^2 \int \int h^*(x, y) h^*(x, z) R(y-z) dy dz \\ &\quad + v / h^*(x, y) R(0) dy - 2v / h^*(x, y) R(x-y) dy + R(0) \\ &= v / h^*(x, y) R(x-y) dy - 2v / h^*(x, y) R(x-y) dy + R(0) \\ &\quad \text{[by (5.3)]} \\ &= h^*(x, x) R(0) \end{aligned}$$

by yet another application of (5.3). \square

kriging" of Matheron (1976). Solution of (5.9) requires more than just the covariance function of Y , but not full knowledge of the distribution: only bivariate distributions appear in (5.9), which is usable if these can be calculated.

The optimal estimators

$$(5.5) \quad Y(x) = \int_A h(x, z) Y(z) W(dz)$$

have one glaring shortcoming, engendered by our using an averaged (via the expectation) error criterion: if $x \in A$ is a point x_1 of M , then even though $Y(x) = Y(x_1)$ is known exactly, (5.5) does not produce $\hat{Y}(x) = Y(x)$. This occurs, of course, because $P(\hat{Y}(x)) \neq 0$, but even so is unsatisfactory.

The estimator

$$(5.6) \quad \hat{Y}'(x) = \hat{Y}(x) + \int_A (Y(z) - \hat{Y}(z)) 1(x=z) W(dz),$$

where \hat{Y} is given by (5.5), satisfies $\hat{Y}'(x_1) = Y(x_1)$ for each point $x_1 \in A$. of M and - albeit not linear - is as easily calculated as \hat{Y} . Moreover, since $P(\hat{Y}'(x) - \hat{Y}(x)) = 1$, (5.4) remains valid; thus pointwise behavior is improved without impairing the mean squared error.

Yet another approach is to allow estimators that are linear functionals of the marked point process \tilde{N} , i.e.,

$$(5.7) \quad \hat{Y}(x) = \int_{A \times R} h(x, y, u) \tilde{N}(dy, du),$$

although this fails to alleviate the difficulty. An argument analogous to the proof of Proposition (5.2) identifies the optimal solution h^* .

(5.8) PROPOSITION. The MSEZ linear state estimator of the form (5.7) corresponds to the function h^* satisfying

$$(5.9) \quad \frac{\partial E[Y(x) 1(Y(y) \in (\cdot))] }{\partial P(Y(y) \in (\cdot))} (u) \\ = h^*(x, y, u) + \int_{A \times R} h^*(x, y', u') \frac{\partial P(Y(y') \in du', Y(y) \in (\cdot))}{\partial P(Y(y) \in (\cdot))} (u) dy' . \square$$

There is some resemblance between this procedure and the "disjunctive

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